

PISOT UNITS, SALEM NUMBERS AND HIGHER DIMENSIONAL PROJECTIVE MANIFOLDS WITH PRIMITIVE AUTOMORPHISMS OF POSITIVE ENTROPY

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ABSTRACT. We show that, in any dimension greater than one, there are an abelian variety, a smooth rational variety and a Calabi-Yau manifold, with primitive birational automorphisms of first dynamical degree > 1 . We also show that there are smooth complex projective Calabi-Yau manifolds and smooth rational manifolds, of any even dimension, with primitive biregular automorphisms of positive topological entropy.

1. INTRODUCTION

Throughout this note, we work in the category of projective varieties defined over \mathbb{C} .

The aim of this note is to give some affirmative answers (Theorems 1.7) to the following question 1.1 (1) asked by [Og15, Problem 1.1] and a weaker version (2) in any dimension, for abelian varieties, smooth rational varieties and Calabi-Yau manifolds:

Question 1.1. (1) For each integer $\ell \geq 2$, is there a smooth projective variety of dimension ℓ with a primitive biregular automorphism of positive topological entropy?
 (2) For each integer $\ell \geq 2$, is there a smooth projective variety of dimension ℓ with a primitive birational automorphism of first dynamical degree > 1 ?

Question 1.1 is related to birational geometry, complex dynamics, and also number theory and representation theory as we shall see (cf. Section 3). We recall the complex dynamical notion of topological entropy and closely related notions of dynamical degrees in Section 2, following [Bo73], [Gr03], [Yo87], [DS05], [DN11] and [Tr15]. Here we just recall that if $f \in \text{Aut}(M)$, then f is of positive entropy $h_{\text{top}}(f) > 0$ if and only if the first dynamical degree $d_1(f) > 1$. The notion of primitivity of automorphism, introduced by De-Qi Zhang [Zh09], is purely algebro-geometric:

Definition 1.2. Let M be a smooth projective variety of dimension $\ell \geq 2$ and $f \in \text{Bir}(M)$. f is called *imprimitive* if there are a dominant rational map $\pi : M \dashrightarrow B$ with connected fibers to a smooth projective variety B with $0 < \dim B < \dim M$, and a rational map $f_B : B \dashrightarrow B$, necessarily $f_B \in \text{Bir}(B)$, such that $\pi \circ f = f_B \circ \pi$. Here smoothness assumption is harmless by resolution of singularities [Hi64], as we work over \mathbb{C} . f is *primitive* if it is not imprimitive.

In the definition, π and f_B are not regular in general even if $f \in \text{Aut}(M)$. If M is a curve, i.e., if $\ell = 1$, then any automorphism of M is primitive and of null topological entropy. So,

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from now, we assume that $\ell \geq 2$. A primitive birational automorphism is in some sense an *irreducible* automorphism in birational geometry. Compare with the following trivial (but related, see the proof of Theorem 8.1):

Remark 1.3. Let k be a field and V be a k -vector space of $\dim V = \ell \geq 1$. Let $f \in \mathrm{GL}(V)$. Then f is not irreducible in the usual sense if and only if there are a k -vector space W with $0 < \dim W < \dim V$, a surjective linear map $\pi : V \rightarrow W$ and $f_W \in \mathrm{GL}(W)$ such that $\pi \circ f = f_W \circ \pi$. The existence of irreducible (V, f) with $\dim V = \ell$ depends on the field k . For instance $\ell = 1$ if $k = \mathbb{C}$, $\ell = 1, 2$ if $k = \mathbb{R}$, while ℓ is arbitrary if $k = \mathbb{Q}$.

The existence of a primitive automorphism also depends on the classes of smooth projective varieties. Indeed, as again observed by De-Qi Zhang [Zh09] (see also [Og15, Theorem 2.2], [Og16, Lemma 3.2, Theorem 3.3] for the formulation here), we have the following rather strong constraint on projective varieties admitting primitive birational automorphisms:

Theorem 1.4. *Let M be a smooth projective variety of dimension $\ell \geq 2$. We denote the Kodaira dimension of M by $\kappa(M)$ and the irregularity of M by $q(M)$. Then, if M has a primitive birational automorphism $f \in \mathrm{Bir}(M)$, then $\mathrm{ord} f = \infty$ and M falls into one of the following three exclusive classes:*

(WRC) $\kappa(M) = -\infty$ and $q(M) = 0$;

(WCY) $\kappa(M) = 0$ and $q(M) = 0$;

(A) M is birational to an abelian variety.

Assume in addition that the following two statements hold for M :

(1) If $\kappa(M) = 0$ and $h^1(\mathcal{O}_M) = 0$, then M is birational to a minimal Calabi-Yau variety M' , i.e., a normal projective variety M' with only \mathbb{Q} -factorial terminal singularities such that $\mathcal{O}_{M'}(mK_{M'}) \simeq \mathcal{O}_{M'}$ for some $m > 0$; and

(2) If $\kappa(M) = -\infty$, then M is uniruled.

Then, M falls into one of the following three exclusive classes:

(RC) M is a rationally connected manifold, i.e., a smooth projective variety whose two general closed points are connected by a rational curve on M ;

(CY) M is birational to a minimal Calabi-Yau variety; or

(A) M is birational to an abelian variety.

Remark 1.5. The assumptions (1) and (2) hold if the minimal model problem and the abundance conjecture are affirmative in dimension ℓ , so that they hold if $\ell \leq 3$ by the minimal model theory and abundance theorem for projective threefolds due to Kawamata, Miyaoka, Mori and Reid ([Mo88], [Ka92], see also [KMM87], [KM98]).

Remark 1.6. Obviously, smooth rational/unirational varieties are rationally connected. The most important classes of smooth projective varieties in (CY) or (WCY) are *Calabi-Yau manifolds* and projective *hyperkähler manifolds* (see eg. [GHJ03] for these manifolds). Recall that an ℓ -dimensional simply-connected smooth projective variety M is a Calabi-Yau manifold (resp. hyperkähler manifold) if $H^0(\Omega_M^j) = 0$ for $0 < j < \ell$ and $H^0(\Omega_M^\ell) = \mathbb{C}\omega_M$ for a nowhere vanishing regular ℓ -form ω_M (resp. $H^0(\Omega_M^2) = \mathbb{C}\eta_M$ for an everywhere non-degenerate regular 2-form η_M , and therefore ℓ is even).

When $\ell = 2, 3$, there are smooth projective rational varieties, Calabi-Yau manifolds and abelian varieties, with primitive biregular automorphisms of positive entropy (See eg.

[Ca99], [BK09], [BK12], [Mc07], [Mc16], [Re12], [CO15], [Do16] for surfaces in several classes and [OT14], [OT15] for threefolds).

Our main result is the following:

Theorem 1.7. (1) *For each $\ell \geq 2$, there is an ℓ -dimensional abelian variety A with a primitive biregular automorphism $f \in \text{Aut}(A)$ of positive topological entropy. There is also an ℓ -dimensional smooth projective variety M , birational to a minimal Calabi-Yau variety, with a primitive biregular automorphism of positive topological entropy.*

(2) *For each $\ell \geq 2$, there is a primitive birational automorphism $f \in \text{Bir}(\mathbb{P}^\ell)$ of first dynamical degree $d_1(f) > 1$.*

(3) *For each $n \geq 2$, there is a $2n$ -dimensional smooth rational variety M with a primitive biregular automorphism $f \in \text{Aut}(M)$ of positive topological entropy.*

(4) *For each $\ell \geq 2$, there is an ℓ -dimensional smooth Calabi-Yau manifold M with a primitive birational automorphism $f \in \text{Bir}(M)$ of first dynamical degree $d_1(f) > 1$.*

(5) *For each $n \geq 2$, there is a $2n$ -dimensional smooth Calabi-Yau manifold M with a primitive biregular automorphism $f \in \text{Aut}(M)$ of positive topological entropy.*

We construct A in (1) as the self product of an elliptic curve E and its desired automorphism by using *Pisot units* (Definition 3.1) in Theorem 4.2. M in (1) is obtained by a standard resolution of the quotient variety $A/\langle -1_A \rangle$ (Corollary 4.3, see also Remark 4.5). Primitivity is checked by the product formula for the relative dynamical degrees (see Section 2 for summary). We also show the existence of rationally connected manifolds of dimension 4 and 5 with primitive biregular automorphisms of positive topological entropy as a byproduct of the existence of certain automorphisms (Corollary 4.6 and the proof). We find a desired birational automorphism in (2) among monomial birational maps studied by [FW12] and [Li12]. Our proof of primitivity is similar to the one for (1) (Theorem 6.5). Our Calabi-Yau manifolds in (4) are Calabi-Yau manifolds of Wehler type studied in [CO15] when ℓ is odd (Theorem 8.1). See also [Og16-2] for a different construction. We construct desired manifolds and their biregular automorphisms in (3) and (5) using K3 surfaces S with special automorphisms and their Hilbert schemes of n points $S^{[n]}$ (Theorems 6.1, 7.1) and check the primitivity by reducing to the following remarkable result due to Bianco [Bi16, Main Theorem]:

Theorem 1.8. *Let M be a hyperkähler manifold and $f \in \text{Bir}(M)$ and $d_1(f)$ the first dynamical degree of f . If $d_1(f) > 1$, then f is primitive.*

It might be worth mentioning that the following question remains open:

Question 1.9. (1) Are there Calabi-Yau manifolds, of *odd* dimension ≥ 5 , with primitive biregular automorphisms of positive topological entropy?

(2) Are there smooth rationally connected manifolds of *odd* dimension ≥ 7 , with primitive biregular automorphisms of positive topological entropy?

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2. ENTROPY, DYNAMICAL DEGREES AND RELATIVE DYNAMICAL DEGREES

In this section we briefly recall definitions of topological entropy, dynamical degrees, relative dynamical degrees and their basic properties, used in this paper. No new result is included. Main references here are Dinh-Sibony [DS05], Dinh-Nguyen [DN11], Dinh-Nguyen-Truong [DNT12] and Truong [Tr15], [Tr16]. We state the results for birational automorphisms of smooth projective varieties, while all the results below are valid for any compact Kähler manifolds and dominant meromorphic selfmaps (if we use Kähler forms instead of Fubini-Study forms or hyperplane classes).

2.1. Topological entropy and Gromov-Yomdin theorem. Let $M = (M, d)$ be a compact metric space and $f : M \rightarrow M$ be a continuous surjective selfmap of M . The *topological entropy* of f is the fundamental invariant that measures *how fast two general points spread out under the action of the semi-group* $\{f^n | n \in \mathbf{Z}_{\geq 0}\}$. For the definition, we define the new distance $d_{f,n}$ on X , depending on f and n , by

$$d_{f,n}(x, y) = \max_{0 \leq j \leq n-1} d(f^j(x), f^j(y)) \text{ for } x, y \in X.$$

Let $\epsilon > 0$ be a positive real number. We call two points $x, y \in M$ (n, ϵ) -separated if $d_{f,n}(y, x) \geq \epsilon$, and a subset $F \subset M$ (n, ϵ) -separated if any two distinct points of F are (n, ϵ) -separated. Let

$$N_d(f, n, \epsilon) := \text{Max} \{ |F| \mid F \subset M \text{ is } (n, \epsilon) \text{ - separated} \} .$$

Note that $N_d(f, n, \epsilon)$ is a well-defined positive integer, as M is compact. The following definition was introduced by Bowen [Bo73]:

Definition 2.1. The *topological entropy* of f is defined by:

$$h_{\text{top}}(f) := \lim_{\epsilon \rightarrow +0} \limsup_{n \rightarrow \infty} \frac{\log N_d(f, n, \epsilon)}{n} .$$

Note that $h_{\text{top}}(f)$ does not depend on the metric d of the topological space M .

Let M be a smooth projective variety and $f : M \rightarrow M$ be a surjective morphism. Then f is continuous in the classical metric topology given by the Fubini-Study metric $d(*, **)$ under any embedding $M \subset \mathbb{P}^N$. So, one can speak of the topological entropy of f . One of the most fundamental properties of the topological entropy is the following cohomological characterization due to Gromov and Yomdin ([Gr03], [Yo87]):

Theorem 2.2. *Let M be a smooth projective variety of dimension ℓ and $f : M \rightarrow M$ be a surjective morphism. Then, $d_p(f) = r_p(f)$ and*

$$h_{\text{top}}(f) = \log \max_{0 \leq p \leq \ell} d_p(f) = \log \max_{0 \leq p \leq \ell} r_p(f) = \log r(f) .$$

Moreover, $h_{\text{top}}(f) > 0$ if and only if $d_1(f) > 1$.

Here $r_p(f)$ and $r(f)$ are the spectral radii of $f^*|H^{p,p}(X)$ and $f^*|\oplus_{p=0}^{2\ell} H^p(M, \mathbf{Z})$ respectively. The p -th dynamical degree $\lambda_p(f)$ is defined (and well-defined) by:

$$d_p(f) = \lim_{n \rightarrow \infty} \left(\int_X (f^n)^*(\omega^p) \wedge \omega^{k-p} \right)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} ((f^n)^*(H^p) \cdot H^{\ell-p})_M^{\frac{1}{n}} \geq 1.$$

Here H is the hyperplane class of M (under any embedding $M \subset \mathbb{P}^N$) and ω is the Fubini-Study form, i.e., the positive closed $(1,1)$ -form induced by the Fubini-Study form of \mathbb{P}^N .

Remark 2.3. (1) The equality $d_p(f) = r_p(f)$ shows that one can replace $H^{p,p}(M)$ by $N^p(M)$, provided that M is projective (see eg. [Tr15], [Tr16]). Here $N^p(M)$ is the finitely generated free \mathbb{Z} -module consisting of the numerical equivalence classes of algebraic p -cocycles.

(2) The first equality $d_p(f) = r_p(f)$ is purely linear algebraic and indeed just follows from $(f^n)^* = (f^*)^n$ for all $n \geq 0$, which is trivial for automorphisms. So, for instance, if $f \in \text{Bir}(M)$ is isomorphic in codimension one, then $d_1(f) = r_1(f)$, as $(f^n)^* = (f^*)^n$ on $N^1(M)$.

2.2. Dynamical degrees. One can define $r_p(f)$, $r(f)$ and $d_p(f)$ also for birational self map $f : M \dashrightarrow M$, i.e., for $f \in \text{Bir}(M)$. Precisely, for each $n \geq 1$, we define the pull back $(f^n)^*$ by $(f^n)^*(a) := (p_{1,n})_*((p_{2,n})^*(a))$ and $d_p(f)$ by the same formula above. Here $p_{1,n} : \tilde{M}_n \rightarrow M$ and $p_{2,n} : \tilde{M}_n \rightarrow M$ be any birational *morphisms* from a smooth projective variety \tilde{M}_n , depending on f and n , such that $f^n = p_{2,n} \circ (p_{1,n})^{-1}$. In general $r_p(f)$ and $r(f)$ (we may just take $n = 1$) are not birational invariant, as the standard Cremona involution of \mathbb{P}^2 , which is not isomorphic in codimension one, shows. On the other hand, the dynamical degrees $d_p(f)$ are well-defined birational invariant, as Dinh-Sibony ([DS05]) proved (See also [Tr15] for purely algebro-geometric proof):

Theorem 2.4. *Let M and M' be smooth projective varieties and $\mu : M \dashrightarrow M'$ be a birational map. Let $f \in \text{Bir}(M)$ and $f' = \mu \circ f \circ \mu^{-1}$ the birational automorphism of M' induced by f and μ . Then, $d_p(f)$ is well defined and $d_p(f) = d_p(f')$ for all p .*

2.3. Relative dynamical degrees.

Set-up 2.5. Let M and B be smooth projective varieties of dimension ℓ and b respectively, $f \in \text{Bir}(M)$ and $\pi : M \rightarrow B$ be an f -equivariant surjective morphism so that there is $f_B \in \text{Bir}(B)$ with $\pi \circ f = f_B \circ \pi$. Here we do not require that π has connected fibers. Let H_M and H_B be the hyperplane class of M and B (under any embedding to projective spaces) and ω_M and ω_B be the Fubini-Study forms induced by the embeddings.

Remark 2.6. If $f \in \text{Bir}(M)$ is imprimitive and $\pi : M \dashrightarrow B$ be f -equivariant, then by replacing M by \tilde{M} , a resolution $\mu : \tilde{M} \rightarrow M$ of indeterminacy of π , f by $\tilde{f} = \mu^{-1} \circ f \circ \mu$ and π by the induced morphism $\pi \circ \mu$, we always obtain \tilde{f} -equivariant surjective morphism $\pi \circ \mu : \tilde{M} \rightarrow B$ as in Set-up 2.5. In this modification, we may loose biregularity of \tilde{f} in general even if $f \in \text{Aut}(M)$ but the values $d_p(f)$ remain the same, i.e., $d_p(\tilde{f}) = d_p(f)$ by Theorem 2.4. So, when we compute the dynamical degree $d_p(f)$, we may assume π is a surjective *morphism*.

The notion of the relative dynamical degrees $d_p(f|\pi)$ is defined by Dinh-Nguyen [DN11] as follows:

Definition 2.7. Under Set-up 2.5, the *relative p -th dynamical degree* $d_p(f|\pi)$ is defined as

$$\lim_{n \rightarrow \infty} \left(\int_M (f^n)^*(\omega_M^p) \wedge \omega_M^{\ell-b-p} \wedge \pi^*(\omega_B^b) \right)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} ((f^n)^*(H_M^p) \cdot H_M^{\ell-b-p} \cdot \pi^*(H_B^b))^{\frac{1}{n}} \geq 1 .$$

$d_p(f|\pi)$ is well-defined by [DN11]. We may regard $\pi^*(H_B^b)$ as a general fiber class of π , up to positive constant multiple. So, the relative dynamical degree could be considered as the dynamical degree of f *restricted to a general fiber F* .

The following important result, called the *product formula*, was proved by Dinh-Nguyen [DN11] (See also [Tr15] for purely algebro-geometric approach):

Theorem 2.8. Under Set-up 2.5, $d_p(f) = \max_{\max\{0, p-\ell+b\} \leq j \leq \min\{p, b\}} d_j(g) d_{p-j}(f|\pi)$.

The following conclusions (1) and (2) of Theorem 2.8 are observed by [DN11, Corollary 1.2] and [OT15, Theorem 4.1 (1)] respectively, and will be used in the proof of our main result:

Corollary 2.9. Let M and B be smooth projective varieties, $f \in \text{Bir}(M)$ be a birational automorphism of M , $\pi : M \dashrightarrow B$ be an f -equivariant dominant rational map and f_B be the induced birational automorphism of B . Then:

- (1) If π is generically finite, i.e., if $\dim M = \dim B$, then $d_p(f) = d_p(f_B)$ for every p .
- (2) If $0 < \dim B < \dim M$, then $d_1(f) \leq d_2(f)$. In particular, f is primitive if $d_1(f) > d_2(f)$.

3. PISOTS UNITS AND SALEM NUMBERS.

In this section, we briefly recall definitions and properties of Pisot units and Salem numbers, which we will use. Theorem 3.6 is a slight generalization of [EOY16, Theorem 4.1] which is used in the proof of Theorem 8.1 and will be applicable in other situation.

First, we recall the definition of Pisot unit and properties we will use.

Definition 3.1. Let $\overline{\mathbb{Z}} \subset \mathbb{C}$ be the ring of algebraic integers. A real algebraic integer $\alpha \in \overline{\mathbb{Z}} \cap \mathbb{R}$ is called a *Pisot number* if $|\alpha| > 1$ and $|\alpha'| < 1$ for all Galois conjugates $\alpha' \neq \alpha$ of α over \mathbb{Q} . Here and hereafter, for a complex number a , we denote the complex conjugate of a by \bar{a} and define $|a| := \sqrt{a\bar{a}}$. The degree of the Pisot number α is the degree of the minimal polynomial of α over \mathbb{Z} or equivalently over \mathbb{Q} . A Pisot number α of degree d is called a *Pisot unit* if α is invertible in $\overline{\mathbb{Z}}$, i.e. $\prod_{j=1}^d \alpha_j = \pm 1$, where α_i ($1 \leq j \leq d$) are the Galois conjugates of α . We note that if α is a Pisot number or a Pisot unit of degree d then so is $-\alpha$. So, *from now on, we may and will assume that Pisot numbers are greater than 1*.

Example 3.2. (1) Let a be an integer such that $a \geq 3$. Then, by definition, the largest root of $X^2 - aX + 1 = 0$ is a Pisot unit of degree 2. Similarly, for a positive integer b , the largest root of $X^2 - bX - 1 = 0$ is a Pisot unit of degree 2.

(2) The largest real roots $\alpha_3 = 1.324\dots$, $\alpha_4 = 1.380\dots$, $\alpha_5 = 1.443\dots$ of the equations $X^3 - X - 1 = 0$, $X^4 - X^3 - 1 = 0$, $X^5 - X^4 - X^3 + X^2 - 1 = 0$ are Pisot units of degree 3, 4, 5 respectively. These three Pisot units $\alpha_3 < \alpha_4 < \alpha_5$ are the smallest three positive Pisot numbers ([BDGPS92, Theorem 7.2.1]).

There are plenty of Pisot units ([BDGPS92, Theorem 5.2.2]):

Theorem 3.3. *Let $d \geq 2$ be any positive integer and K be any real field extension of \mathbb{Q} of degree $d = [K : \mathbb{Q}]$ (for instance $K = \mathbb{Q}(\sqrt[d]{2})$). Then there is a Pisot unit $\alpha \in K$ of degree $d = [K : \mathbb{Q}]$.*

The first dynamical degrees of birational automorphisms of rational surfaces, which are not conjugate to biregular automorphisms, are necessarily Pisot numbers ([DF01], [BC13]). They are not necessarily Pisot units (cf. Section 4).

Next we recall the definition of Salem number (including quadratic units) and properties we will use.

Definition 3.4. A polynomial $P(X) \in \mathbb{Z}[X]$ is called a *Salem polynomial* if it is irreducible over \mathbb{Z} , monic, of even degree $2d \geq 2$ and the complex zeroes of $P(x)$ are of the form $(1 \leq i \leq d-1)$:

$$a > 1, \quad 0 < \frac{1}{a} < 1, \quad \alpha_i, \quad \alpha_{i+d-1} := \bar{\alpha}_i \in S^1 := \{z \in \mathbb{C} \mid |z| = 1\} \setminus \{\pm 1\}.$$

A *Salem number* is the largest real root $a > 1$ of a Salem polynomial $P(X)$ and we call the degree of $P(X)$ the degree of a . By definition, Salem numbers are always units, i.e., in \mathbb{Z}^\times , and of even degree.

Example 3.5. Unlike Pisot numbers, it is unknown which is the smallest Salem number. The smallest known Salem number is the *Lehmer number* $\lambda_{\text{Lehmer}} = 1.17628\dots$, the real root > 1 of the following Salem polynomial of degree 10:

$$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1.$$

It is conjectured that the Lehmer number is the smallest Salem number and in fact it is the smallest one in degree ≤ 40 (see eg. the webpage [Mo03]). Recall that the first dynamical degree $d_1(f)$ of a smooth surface automorphism f is Salem number if $d_1(f) > 1$ (cf. Proposition 3.6). McMullen [Mc07, Theorem A.1] shows that the Lehmer number is the smallest Salem number among the first dynamical degrees $d_1(f) \neq 1$ of smooth surface automorphisms f ([Mc07, Theorem A.1]). He also shows that the Lehmer number is realized as $d_1(f)$ of a rational surface automorphism f and a projective K3 surface automorphism f ([Mc07], [Mc16]).

We have the following purely lattice theoretic result (for a smooth projective surface S , we can apply Theorem 3.6 (1) for $L = N^1(S)$):

Theorem 3.6. *Let $L = (\mathbb{Z}^n, (*, **))$ be a hyperbolic lattice of rank n , i.e. a free \mathbb{Z} -module of rank n with an integral non-degenerate bilinear form $(*, **)$ of signature $(1, n-1)$. Let $L_{\mathbb{R}} = L \otimes \mathbb{R}$ and let $O(L)$ (resp. $O(L_{\mathbb{R}})$) be the orthogonal groups of the hyperbolic lattice L (resp. of the real hyperbolic space $L_{\mathbb{R}}$) with respect to $(*, **)$. We also define*

$$O^+(L_{\mathbb{R}}) := \{f \in O(L_{\mathbb{R}}) \mid f(P) = P\}.$$

Here P is one of the two connected components of $\{x \in L_{\mathbb{R}} \mid (x, x) > 0\}$. Then:

- (1) *Let $f \in O(L) \cap O^+(L_{\mathbb{R}})$. Then the characteristic polynomial $\Phi_f(x) \in \mathbb{Z}[x]$ of f is the product of cyclotomic polynomials (possibly empty) and at most one Salem polynomial (possibly empty), counted with multiplicities.*

- (2) Assume that the rank n of L is even. Let G be a subgroup of $O(L)$. Assume that the action of G on $L_{\mathbb{R}}$ is irreducible, i.e., there is no G -stable linear subspace of $L_{\mathbb{R}}$ other than $\{0\}$ and $L_{\mathbb{R}}$. Then, there is $f \in G$ whose characteristic polynomial $\Phi_f(x)$ is a Salem polynomial of degree n .

Proof. The assertion (1) is well-known, see eg. [Og06, Proposition 2.5] for a self-contained proof.

We shall prove the assertion (2). Note that $O(L_{\mathbb{R}})$ is a real algebraic group and the special real orthogonal group $SO(L_{\mathbb{R}})$ is an algebraic subgroup of $O(L_{\mathbb{R}})$ of index two. Thus by [BH04, Proposition 1], the Zariski closure of G in $O(L_{\mathbb{R}})$ is either $SO(L_{\mathbb{R}})$ or $O(L_{\mathbb{R}})$. We also note that $g^2 \in SO^+(L_{\mathbb{R}}) := O^+(L_{\mathbb{R}}) \cap SO(L_{\mathbb{R}})$ for $g \in O(L_{\mathbb{R}})$. Let $P_n \simeq \mathbb{R}^n$ be the real affine space consisting of the real monic polynomial of degree n . Then, as n is even, the same proof in [EOY16, Theorem 4.1] applied for the real algebraic morphism

$$G \rightarrow P_n ; g \mapsto \Phi_{g^2}(x)$$

concludes that there is $f \in G$ such that the characteristic polynomial $\Phi_{f^2}(x)$ of f^2 is a Salem polynomial of degree n . \square

4. ABELIAN VARIETIES WITH PRIMITIVE AUTOMORPHISMS OF POSITIVE TOPOLOGICAL ENTROPY AND A FEW APPLICATIONS.

Our main results of this section are Theorems 3.3, 4.1 and Corollaries 4.3, 4.6, from which Theorem 1.7 (1) follows.

Pisot numbers, or more precisely Pisot units, play important roles in primitive automorphisms of abelian varieties:

Theorem 4.1. *Let d be an integer such that $d \geq 2$. Let A be a d -dimensional abelian variety and $f \in \text{Aut}(A)$. Assume that $f^*|H^0(A, \Omega_A^1)$ has a Pisot number $\alpha > 1$ of degree d as its eigenvalue. Then α is a Pisot unit and f is a primitive automorphism of A of positive topological entropy $h_{\text{top}}(f) = 2 \log \alpha > 0$.*

Proof. By the Hodge decomposition theorem, we have

$$H^1(A, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H^1(A, \mathbb{C}) = H^0(A, \Omega_A^1) \oplus \overline{H^0(A, \Omega_A^1)}.$$

This decomposition is compatible with the action of f^* . Thus the eigenvalues of $f^*|H^1(A, \mathbb{C})$ are $\alpha_j, \bar{\alpha}_j$ ($1 \leq j \leq d$), counted with multiplicities. Here $\alpha_1 := \alpha$ and α_j ($1 \leq j \leq d$) are the Galois conjugates of α . As f is an automorphism, $f^*|H^1(A, \mathbb{Z})$ is invertible over \mathbb{Z} . Thus the determinant of $f^*|H^1(A, \mathbb{Z})$, which is $\pm \prod_j |\alpha_j|^2$, is ± 1 . Hence $\prod_j \alpha_j$ is ± 1 as $\prod_j \alpha_j \in \mathbb{Z}$. This shows that α is a Pisot unit. As $\alpha = \alpha_1$ is a positive Pisot number, we may order α_j so that

$$|\alpha_1| = \alpha > 1 > |\alpha_2| \geq \dots \geq |\alpha_d|.$$

As A is an abelian variety, we have

$$H^{p,p}(A) = \wedge^p H^0(A, \Omega_A^1) \otimes_{\mathbb{C}} \overline{\wedge^p H^0(A, \Omega_A^1)}.$$

It follows that

$$d_k(f) = \prod_{j=1}^k |\alpha_j|^2.$$

In particular,

$$d_1(f) = |\alpha_1|^2, \quad d_2(f) = |\alpha_1|^2 |\alpha_2|^2, \quad d_3(f) = |\alpha_1|^2 |\alpha_2|^2 |\alpha_3|^2, \quad \dots$$

and therefore, $h_{\text{top}}(f) = 2 \log \alpha > 0$ by $\alpha = |\alpha_1| > 1$ and $|\alpha_j| < 1$ for $j \geq 2$ and by Theorem 2.2.

As $|\alpha_2| < 1$, it follows that $d_1(f) > d_2(f)$. Hence f is primitive by Corollary 2.9 (2). This completes the proof of Theorem 4.1. \square

Theorem 4.2. *Let d be an integer such that $d \geq 2$. Let $\alpha > 1$ be any Pisot unit of degree d , whose existence is guaranteed by Theorem 3.3. Let E be any elliptic curve. Then the d -dimensional abelian variety E^d admits a primitive automorphism $f \in \text{Aut}_{\text{group}}(E^d)$ with*

$$h_{\text{top}}(f) = 2 \log \alpha > 0.$$

Proof. Let $S_d(X) = X^d + a_d X^{d-1} + \dots + a_2 X + a_1 \in \mathbb{Z}[X]$ be the minimal polynomial of α . Note that $a_1 = \pm 1$ as α is a Pisot unit. Consider the matrix $M_d = (m_{ij}) \in M_d(\mathbb{Z})$ whose entries m_{ij} are 0 except

$$m_{dj} = -a_j, \quad m_{i,i+1} = 1 \quad (1 \leq j \leq d, \quad 1 \leq i \leq d-1),$$

associated to the polynomial $S_d(X)$. For instance

$$M_2 = \begin{pmatrix} 0 & 1 \\ -a_1 & -a_2 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 & -a_4 \end{pmatrix}.$$

By definition of M_d , the characteristic polynomial of M_d is $S_d(X)$ and, as $a_1 = \pm 1$, we have $M_d \in \text{GL}_d(\mathbb{Z})$. Let (z_1, z_2, \dots, z_d) be the standard complex coordinates of the universal cover \mathbb{C}^d of E^d . Then, as $M_d \in \text{GL}_d(\mathbb{Z})$, we can uniquely define the group automorphism $f \in \text{Aut}_{\text{group}}(E^d)$ by

$$f^*(z_1, z_2, \dots, z_d)^t = M_d(z_1, z_2, \dots, z_d)^t.$$

Here t is the transpose. Then $f^*|H^0(E^d, \Omega_{E^d}^1) = M_d$ under the basis $\langle dz_i \rangle_{i=1}^d$ of $H^0(E^d, \Omega_{E^d}^1)$. The characteristic polynomial of $f^*|H^0(E^d, \Omega_{E^d}^1)$ is then $S_d(X)$, the minimal polynomial of Pisot unit α of degree d . The result now follows from Theorem 4.1. \square

The first part of Theorem 1.7 (1) follows from Theorem 4.2.

Corollary 4.3. *Let d be an integer such that $d \geq 3$. Let $\alpha > 1$ be any Pisot unit of degree d , whose existence is guaranteed by Theorem 3.3. Then there is a d -dimensional smooth projective variety M , birational to a minimal Calabi-Yau variety, such that M admits a primitive automorphism f with $h_{\text{top}}(f) = 2 \log \alpha > 0$.*

Proof. Let M be the blow up at the maximal ideals of the singular points of the quotient variety $\overline{M} := E^d / \langle -1_{E^d} \rangle$. Then \overline{M} is a minimal Calabi-Yau variety as $d \geq 3$, and M is a smooth projective variety birational to \overline{M} . The automorphism $f \in \text{Aut}_{\text{group}}(E^d)$ in Theorem 4.1 descends to an automorphism $f_M \in \text{Aut}(M)$ of M as $f \circ (-1_{E^d}) = (-1_{E^d}) \circ f$ and by the universality of the blow up. By Corollary 2.9 (1), $d_p(f_M) = d_p(f)$ for all p . Thus, by Theorem 2.2, $h_{\text{top}}(f_M) = h_{\text{top}}(f)$, which is $2 \log \alpha$ by Theorem 4.2. f_M is also primitive by Lemma 4.4 below. \square

Lemma 4.4. *Let U and V be smooth projective varieties of the same dimension d and $\mu : U \dashrightarrow V$ be a dominant rational map (necessarily generically finite). Let $f_U \in \text{Bir}(U)$ and $f_V \in \text{Bir}(V)$. Then, f_V is primitive if f_U is primitive and $\mu \circ f_U = f_V \circ \mu$.*

Proof. If $\pi : V \dashrightarrow B$ is an f_V -equivariant dominant rational map of connected fibers, then the Stein factorization of $\pi \circ \mu : U \dashrightarrow B$ is an f_U -equivariant dominant rational map of connected fibers, with $(\dim B)$ -dimensional base space. As f_U is primitive and $\dim V = \dim U = d$, it follows $\dim B = 0$ or $\dim B = d$. Hence f_V is primitive. \square

Corollary 4.3 shows the second half part of Theorem 1.7 (1).

Remark 4.5. The manifolds M constructed in the proof of Corollary 4.3 are not birational to a Calabi-Yau manifold nor a hyperkähler manifold when $\dim M \geq 3$. The reason is as follows. Notice that \overline{M} is birational to M , has only isolated terminal singularities with numerically trivial canonical divisor and $\pi_1(\overline{M} \setminus S)$ is an infinite group for any proper closed algebraic subset S such that $\text{Sing } \overline{M} \subset S$. On the other hand, if \overline{M} would be birational to a Calabi-Yau manifold or a hyperkähler manifold, then they would be isomorphic in codimension one (see eg. [Ka08]) so that $\pi_1(\overline{M} \setminus S) = \{1\}$ for some S above of codimension ≥ 2 , a contradiction.

Corollary 4.6. *Let α_d ($d = 3, 4, 5$) be the first three smallest positive Pisot units of degree d respectively (Example 3.2). Then the logarithm $2 \log \alpha_d$ ($d = 3, 4, 5$) is realized as the topological entropy of a primitive automorphism of an abelian varieties of dimension $d = 3, 4, 5$ respectively. Moreover, $2 \log \alpha_3$ is also realized as the topological entropy of a primitive automorphism of a 3-dimensional Calabi-Yau manifold and a 3-dimensional smooth rational variety, and $2 \log \alpha_d$ ($d = 4, 5$) is also realized as the topological entropy of a primitive automorphism of a d -dimensional smooth rationally connected variety R_d .*

Proof. In Theorem 4.1, we choose E to be $E_\omega = \mathbb{C}/\mathbb{Z} + \omega\mathbb{Z}$ ($\omega = (-1 + \sqrt{-3})/2$). Then $f_d := f \in \text{Aut}_{\text{group}}(E_\omega^d)$ in Theorem 4.1, associated to α_d ($d = 3, 4, 5$), satisfies the first requirement.

Let $\mu : R_d \rightarrow \overline{R}_d$ be the blow up at the maximal ideals of the singular points of the quotient variety $\overline{R}_d := E_\omega^d / \langle -\omega I_d \rangle$. Then R_d is a smooth projective variety. Moreover, the automorphism $f_d \in \text{Aut}_{\text{group}}(E_\omega^d)$ descends to an automorphism $f_{R_d} \in \text{Aut}(R_d)$ of R_d and f_{R_d} is primitive of positive topological entropy $2 \log \alpha_d$, exactly for the same reason as in Corollary 4.3.

Let V_3 be the blow up at the maximal ideals of the singular points of the quotient variety $E_\omega^3 / \langle \omega I_3 \rangle$. Then V_3 is a smooth Calabi-Yau threefold and the automorphism $f_{V_3} \in \text{Aut}(V_3)$ of V_3 induced by f_3 satisfies all the required properties.

R_3 is rational by [OT15], R_4 is unirational by [COV15] and R_5 is rationally connected by [KL09, Corollary 25].

Here, we shall give an alternative uniform proof of rational connectedness of R_d ($d = 3, 4, 5$), using the fact that $f_{R_d} \in \text{Aut}(R_d)$ is primitive.

By construction, \overline{R}_d ($d = 3, 4, 5$) has numerically trivial canonical divisor and only isolated singular points which are the image of the fixed points of $\langle -\omega I_3 \rangle$. Let $P \in \overline{R}_d$ be the image of the origin of E_ω^d . As $d \leq 5$, \overline{R}_d is klt but not canonical at P . Let $E \subset R_d$ be the exceptional divisor lying over P . Then we have $K_{R_d} \equiv -aE + E'$ with $a > 0$. Here E' is a divisor whose support lies over the singular points of \overline{R}_d other than P .

Let $y \in R_d$ be a general point of R_d and set $x = \mu(y)$. As y is general, x is a smooth point of \overline{R}_d . Choose an ample divisor H of \overline{R}_d . Then, there is a positive integer m (may depends on y) and a complete intersection curves $\overline{C} = H_1 \cap H_2 \cap \dots \cap H_{d-1}$ ($H_i \in |mH|$) such that \overline{C} is irreducible, $\overline{C} \ni x$, $\overline{C} \ni P$ and \overline{C} contains no other singular points of \overline{R}_d . This is possible, as \overline{R}_d has only finitely many singular points. Let $C \subset R_d$ be the strict transform of \overline{C} . Then $(K_{R_d}.C) = -a(E.C) + (E'.C) = -a(E.C) < 0$. Note that $y \in C$ as $x \in \overline{C}$ and x is a smooth point. Therefore, for a general point $y \in R_d$, there is an irreducible curve $C \subset R_d$ such that $y \in C$ and $(K_{R_d}.C) < 0$. Hence R_d is uniruled by the numerical criterion of the uniruledness due to Miyaoka-Mori ([MM86], see also [Ko96, Chap. IV, Theorem 1. 13]). Here one can also apply [HMZ14, Lemma 2.4] to conclude the uniruledness of R_d ($3 \leq d \leq 5$).

Now consider the maximal rationally connected fibration $\pi : R_d \dashrightarrow B$ of R_d . It is unique up to bitational equivalence ([KMM92], [Ko96, Chap. IV, Theorem 5.5]). In particular, π is f_{R_d} -equivariant. Here B is not uniruled by [GHS03]. Thus $\dim B < \dim R_d$ as R_d is uniruled. As f_{R_d} is primitive, it follows that $\dim B = 0$. Hence R_d is rationally connected. \square

The following question is yet open and is of its own interest (see [COV15] for some attempt):

Question 4.7. Are R_4 and R_5 constructed in our proof of Corollary 4.6 rational?

Note that R_1 and R_2 are obviously rational and R_3 is also rational ([OT15]), while R_d is never rational (even never uniruled) for $d \geq 6$, as $E_\omega^d / \langle -\omega I_d \rangle$ ($d \geq 6$) has only canonical singularities with numerically trivial canonical divisor.

5. HILBERT SCHEMES OF POINTS AND HYPERKÄHLER MANIFOLDS WITH PRIMITIVE AUTOMORPHISMS OF POSITIVE TOPOLOGICAL ENTROPY.

Our main result of this section is Theorem 5.4. Salem numbers also naturally appear as the dynamical degrees of automorphisms of projective hyperkähler manifolds ([Og09]):

Proposition 5.1. *Let M be a (not necessarily projective) hyperkähler manifold of dimension $2m$ and $f \in \text{Aut}(M)$. Then, the p -th dynamical degree $d_p(f)$ is either 1 or a Salem number of degree $\leq b_2(M)$. Moreover, $d_p(f) = d_{2m-p}(f) = d_1(f)^p$, $0 \leq p \leq m$, and $h_{\text{top}}(f) = m \log d_1(f)$.*

Let $S^{[n]} = \text{Hilb}^n(S)$ be the Hilbert scheme of 0-dimensional closed subschemes of lengths n of a smooth projective surface S . Then $S^{[n]}$ is a smooth projective variety of dimension $2n$ by Fogarty [Fo68]. Let $f \in \text{Aut}(S)$. Then f naturally induces an automorphism of $S^{[n]}$, which we denote by $f^{[n]} \in \text{Aut}(S^{[n]})$.

Remark 5.2. It is well known that $S^{[n]}$ is a projective hyperkähler manifold of dimension $2n$ if S is a projective K3 surface ([Fu83] for $n = 2$, [Be83] for arbitrary n). If S is an Enriques surface, then the universal cover M of $S^{[n]}$, which is of covering degree 2, is a Calabi-Yau manifold of dimension $2n$ ([OS11]). If S is a smooth rational surface, then $S^{[n]}$ is a smooth projective rational variety of dimension $2n$. Indeed, $S^{[n]}$ is birational to the symmetric product $\text{Sym}^n(\mathbb{C}^2)$, the later of which is rational by classical invariant theory (see eg. [GKZ94, Chap. 4, Theorem 2.8] for details).

Proposition 5.3. *Let S be a projective K3 surface with an automorphism $f \in \text{Aut}(S)$ such that $d_1(f)$ is a Salem number. Then the automorphism $f^{[n]}$ of $S^{[n]}$ ($n \geq 2$) is primitive and of positive topological entropy.*

Proof. Let E be the exceptional divisor of the Hilbert-Chow morphism $S^{[n]} \rightarrow \text{Sym}^n(S)$. By [Be83], we have an isomorphism $H^2(S^{[n]}, \mathbb{Z}) \simeq H^2(S, \mathbb{Z}) \oplus \mathbb{Z}(E/2)$ compatible with Hodge decomposition and the actions of $f^{[n]}$ and $f \oplus \text{id}_{\mathbb{Z}(E/2)}$. Thus, by Theorem 2.2, $d_1(f^{[n]}) = r_1(f^{[n]}) = r_1(f) = d_1(f)$. Hence $f^{[n]}$ is of positive topological entropy and it is primitive by Theorem 1.8. \square

The following consequence may be of its own interest:

Theorem 5.4. (1) *Let M be a (not necessarily projective) hyperkähler fourfold and $f \in \text{Aut}(M)$. Then $h_{\text{top}}(f) \geq 2 \log \lambda_{\text{Lehmer}}$ unless $h_{\text{top}}(f) = 0$. Here λ_{Lehmer} is the Lehmer number (See Example 3.5).*

(2) *There is a projective hyperkähler fourfold M with a primitive automorphism $f \in \text{Aut}(M)$ of the smallest possible positive topological entropy $h_{\text{top}}(f) = 2 \log \lambda_{\text{Lehmer}}$.*

Proof. By Proposition 5.1, $d_1(f) = 1$ or $d_1(f)$ is a Salem number. Guan [Gu01] shows that $3 \leq b_2(M) \leq 8$ or $b_2(M) = 23$ for hyperkähler fourfolds. $d_1(f)$ is then of degree ≤ 23 again by Proposition 5.1. Thus, as remarked in Example 3.5, it follows from [Mo03] that

$$d_1(f) \geq \lambda_{\text{Lehmer}}, \text{ unless } d_1(f) = 1.$$

Thus, again by Proposition 5.1, $h_{\text{top}}(f) \geq 2 \log \lambda_{\text{Lehmer}}$ unless $h_{\text{top}}(f) = 0$. This proves (1).

As remarked in Example 3.5, McMullen [Mc16] shows that there is a projective K3 surface S with automorphism $f \in \text{Aut}(S)$ such that $d_1(f)$ is the Lehmer number. By Proposition 3.6, $(S^{[2]}, f^{[2]})$ gives then a desired example in (2). \square

In the view of Proposition 5.1 and Theorem 5.4, it is interesting to ask the following:

Question 5.5. Let M be a (not necessarily projective) hyperkähler manifold of dimension $2m \geq 6$ and $f \in \text{Aut}(M)$. Is then $h_{\text{top}}(f) \geq m \log \lambda_{\text{Lehmer}}$ unless $h_{\text{top}}(f) = 0$?

6. SMOOTH RATIONAL MANIFOLDS WITH PRIMITIVE BIRATIONAL AUTOMORPHISMS.

In this section, we shall show Theorems 6.1 and 6.5, from which Theorem 1.7 (2) and (3) follows.

Theorem 6.1. *Let S be a projective K3 surface with $f \in \text{Aut}(S)$ such that $d_1(f) > 1$. Assume that there is $\iota \in \text{Aut}(S)$ such that ι is of finite order, the minimal resolution T of $S/\langle \iota \rangle$ is a rational surface and $f \circ \iota = \iota \circ f$. We denote by f_T the automorphism of T induced by f . Then $T^{[n]}$ is a $2n$ -dimensional smooth rational manifold and the induced automorphism $f_T^{[n]} \in \text{Aut}(T^{[n]})$ is primitive and of positive entropy.*

Example 6.2. Let E and F be mutually non-isogenous elliptic curves and $S := \text{Km}(E \times F)$ be the Kummer K3 surface associated with the product abelian surface $E \times F$. We denote by ω_S a nowhere vanishing holomorphic 2-form on S . Let $\iota \in \text{Aut}(S)$ be the automorphism of S of order 2, induced by $(-1_E, 1_F) \in \text{Aut}(E \times F)$. Note that $\iota^* \omega_S = -\omega_S$ and ι has fixed curves. So, the surface $T = S/\langle \iota \rangle$ is smooth and it is rational by Castelnuovo's criterion. By [Yu16, Remark 6.3], S admits an automorphism f such that $d_1(f)$ is a Salem number

of degree 10. The triple (S, f, ι) then satisfies the condition in Theorem 6.1, as ι is in the center of $\text{Aut}(S)$ by [Og89, Proof of Lemma 1.4].

Proof. $T^{[n]}$ is a smooth projective rational variety of dimension $2n$ (See Remark 5.2). The quotient morphism $S \dashrightarrow T$ induces a dominant rational map $\nu : S^{[n]} \dashrightarrow T^{[n]}$. The map ν is equivariant under the automorphisms $f_S^{[n]}$ and $f_T^{[n]}$. As $d_1(f_S)$ is Salem number, the automorphism $f_S^{[n]}$ is primitive and of positive entropy by Proposition 5.3. Thus so is $f_T^{[n]}$ by Lemma 4.4 and Corollary 2.9 (1). \square

Definition 6.3. Let $(\mathbb{C}^\times)^d$ be the d -dimensional algebraic torus and (t_1, t_2, \dots, t_d) be the standard coordinate of $(\mathbb{C}^\times)^d$. We naturally regard $(\mathbb{C}^\times)^d \subset \mathbb{P}^d$. The birational map $\varphi_A \in \text{Bir}(\mathbb{P}^d)$ ($A = (a_{ij}) \in \text{GL}_d(\mathbb{Z})$) defined by

$$\varphi_A^*(t_1, t_2, \dots, t_d) = (\prod_{j=1}^d t_j^{a_{j1}}, \prod_{j=1}^d t_j^{a_{j2}}, \dots, \prod_{j=1}^d t_j^{a_{jd}})$$

is called the monomial map associated to $A \in \text{GL}_d(\mathbb{Z})$.

Note that $\varphi_{AB} = \varphi_A \circ \varphi_B$ and $\varphi_{A^{-1}} = \varphi_A^{-1}$ for $A, B \in \text{GL}_d(\mathbb{Z})$. The monomial maps are intensively studied by [HP07], [FW12], [Li12] from dynamical point of views. In particular, the closed formula of the dynamical degrees of monomial maps is conjectured by [HP07] and Favre and Wulcan [FW12, Corollary] and Lin [Li12, Theorem 1] confirm the conjecture:

Theorem 6.4. *Let $A \in \text{GL}_d(\mathbb{Z})$ and $\varphi_A \in \text{Bir}(\mathbb{P}^d)$ be the associated monomial map. Order the eigenvalues α_j ($1 \leq j \leq d$) of A as*

$$|\alpha_1| \geq |\alpha_2| \geq \dots \geq |\alpha_d|.$$

Then $d_k(\varphi_A) = \prod_{j=1}^k |\alpha_j|$.

Theorem 6.5. *Let $d \geq 2$ and $\alpha > 1$ be any Pisot unit of degree d (Theorem 3.3). Then \mathbb{P}^d has a primitive birational automorphisms of first dynamical degree $\alpha > 1$.*

Proof. Our proof is quite parallel to the proof of Theorem 3.3. Let $S_d(X) \in \mathbb{Z}[X]$ be the minimal polynomial of α . As in the proof of Theorem 3.3, there is $M \in \text{GL}_d(\mathbb{Z})$ whose characteristic polynomial is $S_d(X)$. Order the complex zeros of $S_d(X)$, i.e., eigenvalues of M as

$$\alpha_1 := \alpha \geq |\alpha_2| \geq \dots \geq |\alpha_d|.$$

Then by Theorem 6.4, the monomial map φ_M satisfies that $d_1(\varphi_M) = \alpha$ and $d_2(\varphi_M) = \alpha|\alpha_2|$. As α is a Pisot unit, we have $\alpha > 1$ and $|\alpha_2| < 1$. Hence,

$$d_1(\varphi_M) = \alpha > 1, \quad d_1(\varphi_M) > d_2(\varphi_M).$$

In particular, φ_M is primitive by the second inequality and Corollary 2.9 (2). This completes the proof of Theorem 6.5. \square

7. EVEN DIMENSIONAL CALABI-YAU MANIFOLDS WITH PRIMITIVE AUTOMORPHISMS OF POSITIVE TOPOLOGICAL ENTROPY.

In this section, we show the following theorem, from which Theorem 1.7 (5) follows:

Theorem 7.1. *For each positive integer n , there is a $2n$ -dimensional Calabi-Yau manifold M with a primitive automorphism $f \in \text{Aut}(M)$ of positive topological entropy.*

Proof. We construct a desired Calabi-Yau manifold from an Enriques surface W described in the following (see [CO15], [Do16] for such Enriques surfaces and automorphisms):

Proposition 7.2. *There is an Enriques surface W with an automorphism $f_W \in \text{Aut}(W)$ such that $d_1(f_W) > 1$.*

In what follows, we choose and fix W and $f_W \in \text{Aut}(W)$ in Proposition 7.2.

Let us consider the Hilbert scheme $W^{[n]}$ and the automorphism $f_W^{[n]} \in \text{Aut}(W^{[n]})$ induced by f_W . Let $u : M \rightarrow W^{[n]}$ be the universal cover of $W^{[n]}$, of covering degree 2. Then:

Proposition 7.3. (1) *M is a $2n$ -dimensional smooth projective Calabi-Yau manifold.*
 (2) *M admits a primitive automorphism of positive topological entropy.*

Proof. The first assertion (1) follows from [OS11], as mentioned in Remark 5.2.

Let $\tau : S \rightarrow W$ be the universal cover of W . Then S is a projective K3 surface and τ is an étale covering of degree 2. As in Theorem 6.1, we reduce the proof of (2) to the hyperkähler manifold $S^{[n]}$.

We denote by $\iota \in \text{Aut}(S)$ the covering involution of τ . Let $f_S \in \text{Aut}(S)$ be any one of the two lifts of $f_W \in \text{Aut}(W)$. We denote by $f_S^{[n]} \in \text{Aut}(S^{[n]})$ the automorphism of $S^{[n]}$ induced by f_S .

Lemma 7.4. *$f_S^{[n]} \in \text{Aut}(S^{[n]})$ is a primitive automorphism of positive topological entropy.*

Proof. As τ is equivariant under f_S and f_W , we have $d_1(f_S) = d_1(f_W)$ by Corollary 2.9 (1). Thus $d_1(f_S)$ is a Salem number. Hence $f_S^{[n]}$ is primitive and of positive topological entropy by Proposition 5.3. \square

An essential point of proof of proposition 7.3 (2) is that $S^{[n]}$, M and $W^{[n]}$ are mutually related under dominant rational maps, as observed by Hayashi [Ha15]. Here we briefly recall some relations from [Ha15, Section 2] with simplifications.

Let V^n (resp. $V^{(n)}$) be the product (resp. the symmetric product) of a smooth projective manifold V and Σ_n be the symmetric group of degree n . Then, by definition, the natural morphism $\mu_V : V^n \rightarrow V^{(n)}$ is a finite Galois cover of Galois group Σ_n and we have also the Hilbert-Chow morphism $\nu_V : V^{[n]} := \text{Hilb}^n(V) \rightarrow V^{(n)}$.

Then, in our situation, we have a natural finite morphism $q : S^n \rightarrow W^{(n)}$, which is Galois whose Galois group is

$$G := \{\iota_I \mid I \subset \{1, 2, \dots, n\}\} \cdot \Sigma_n = \pi_1(W)^n \cdot \Sigma_n \subset \text{Aut}(S^n).$$

Here Σ_n is the group of permutations of the n factors of S^n and $\iota_I \in \text{Aut}(S^n)$ is defined by

$$\iota_I((x_i)_{i=1}^n) = (y_i)_{i=1}^n ; y_i := \iota(x_i) \ (i \in I), \ y_j := x_j \ (j \notin I).$$

$f_{S^n} := (f_S, f_S, \dots, f_S) \in \text{Aut}(S^n)$ commutes with each element of G , as $f_S \circ \iota = \iota \circ f_S$. The group G is of order $2^n \cdot n!$ with a normal subgroup H of index two:

$$H := \{\iota_I \mid |I| \in 2\mathbb{Z}\} \cdot \Sigma_n \subset G.$$

Then we have finite morphisms

$$\bar{v} : S^{(n)} = S^n / \Sigma_n \rightarrow \bar{M} := S^n / H, \ \bar{u} : \bar{M} \rightarrow W^{(n)} = S^n / G.$$

The first morphism is of degree 2^{n-1} , which is not Galois if $n \geq 3$, but the second one is Galois with Galois group $G/H = \langle [\iota_{\{1\}}] \rangle \simeq \mathbb{Z}/2$. As noticed in [OS11, Section 1], we have

$$\pi_1(W^{[n]}) \simeq \pi_1(W^{(n)}) \simeq H_1(W, \mathbb{Z}) = \pi_1(W) = \langle \iota \rangle .$$

Here the first isomorphism is given by the Hilbert-Chow morphism and the second one is given by $[\iota_{\{1\}}] \leftrightarrow \iota$. Thus, \bar{u} is the universal cover of $W^{(n)}$. The universal cover of $W^{[n]}$ is then given by

$$u = \bar{u} \times_{W^{[n]}} \nu_W : M = \bar{M} \times_{W^{(n)}} W^{[n]} \rightarrow W^{[n]} .$$

The following proposition completes the proof of Proposition 7.3 (2):

Proposition 7.5. *At least one of the two lifts, say $f_M \in \text{Aut}(M)$, of $f_{W^{[n]}}$ is primitive and of positive topological entropy.*

Proof. Note that u is equivariant with respect to f_M and $f_{W^{[n]}}$ for any lift f_M of $f_{W^{[n]}}$.

Let $f_{W^{(n)}}$ (resp. $f_{S^{(n)}}$) be the automorphism of $W^{(n)}$ (resp. $S^{(n)}$) induced by $f_W \in \text{Aut}(W)$ (resp. $f_S \in \text{Aut}(S)$), any one of the two lifts of f_W . Then, by the description above, $f_{S^{(n)}}$ descends to $f_{\bar{M}} \in \text{Aut}(\bar{M})$ so that \bar{v} and \bar{u} are equivariant with respect to $f_{S^{(n)}}$, $f_{\bar{M}}$ and $f_{W^{(n)}}$. In particular,

$$\tilde{f}_M := f_{\bar{M}} \times_{f_{W^{(n)}}} f_{W^{[n]}} \in \text{Aut}(M) ,$$

under the description $M = \bar{M} \times_{W^{(n)}} W^{[n]}$ of the universal covering space above, and \tilde{f}_M is one of the two lifts of $f_{W^{[n]}}$ to M . We choose this \tilde{f}_M as f_M . Then the natural rational map

$$q : S^{[n]} \rightarrow \bar{M} \dashrightarrow M$$

is equivariant with respect to $f_{S^{[n]}}$ and f_M . As q is a generically finite dominant rational map and $f_{S^{[n]}}$ is primitive and of positive entropy, so is f_M by Corollary 2.9 (1) and Lemma 4.4. \square

This completes the proof of Proposition 7.3. \square

Theorem 7.1 now follows from Proposition 7.3. \square

Remark 7.6. The universal covering Calabi-Yau $2n$ -folds of the Hilbert scheme $W^{[n]}$ of an Enriques surface W may have other interesting geometric structures. See [Ha15] for some interesting observations.

8. ODD DIMENSIONAL CALABI-YAU MANIFOLDS WITH PRIMITIVE BIRATIONAL AUTOMORPHISMS OF FIRST DYNAMICAL DEGREE > 1 .

Throughout this section we fix an n -dimensional Calabi-Yau manifold of Wehler type M , i.e., a general hypersurface M of multi-degree $(2, 2, \dots, 2)$ in

$$(\mathbb{P}^1)^{n+1} = \mathbb{P}_1^1 \times \dots \times \mathbb{P}_k^1 \times \dots \times \mathbb{P}_{n+1}^1$$

with $n \geq 3$. M is indeed an n -dimensional Calabi-Yau manifold.

Our main result of this section is the following:

Theorem 8.1. *Any odd dimensional Calabi-Yau manifold M of Wehler type has a primitive birational automorphism of first dynamical degree > 1 .*

Theorem 1.7 (4) follows from Theorems 8.1 and 7.1. Our proof of Theorem 8.1 is much inspired by arguments in [Bi16], and is based on Theorem 8.2 below (the main theorem of [CO15]).

For the statement of Theorem 8.2, we recall that

$$N^1(M) \simeq \text{Pic } M = \bigoplus_{k=1}^{n+1} \mathbb{Z} h_k \simeq \mathbb{Z}^{n+1}, \quad \overline{\text{Amp}}(M) = \bigoplus_{k=1}^{n+1} \mathbb{R}_{\geq 0} h_k.$$

Here h_k is the pullback of the hyperplane class of \mathbb{P}_k^1 under the natural projection $M \rightarrow \mathbb{P}_k^1$. As $f : M \dashrightarrow M$ is isomorphic in codimension one for $f \in \text{Bir}(M)$ (see eg. [Ka08], which is also very crucial in the proof of Theorem 8.2), we have a natural contravariant representation

$$\rho : \text{Bir}(M) \rightarrow \text{GL}(N^1(M)) \simeq \text{GL}(\mathbb{Z}^{n+1}); \quad f \mapsto f^*.$$

Let $p_k : M \rightarrow P_k$ ($1 \leq k \leq n+1$) be the natural projection, where P_k is the product manifold $(\mathbb{P}^1)^n$ obtained by removing the k -th factor of $(\mathbb{P}^1)^{n+1}$. The morphism p_k is of degree 2 and the covering involution ι_k of p_k is a (not biregular) birational automorphism of M of order two.

Theorem 8.2. *Let M be a Calabi-Yau manifold of Wehler type of dimension ≥ 3 . Then:*

- (1) $\text{Aut}(M) = \{id\}$, $\text{Bir}(M) = \langle \iota_1, \dots, \iota_{n+1} \rangle$ and $(\text{Bir}(M), \{\iota_k\}_{k=1}^{n+1})$ forms a Coxeter group isomorphic to the universal Coxeter group $UC(n+1)$.
- (2) The representation ρ is faithful and coincides with the dual of the geometric representation of the universal Coxeter group $(\text{Bir}(M), \{\iota_k\}_{k=1}^{n+1}) \simeq UC(n+1)$. In particular, there is an integral non-degenerate bilinear form $(*, **)$ on $N^1(M) \simeq \mathbb{Z}^{n+1}$ of signature $(1, n)$, with respect to which the action of $\text{Bir}(M)$ is orthogonal and ι_k is an orthogonal reflection with respect to the hypersurface $\bigoplus_{j \neq k} \mathbb{R} h_j$ (see [CO15, Section 2.2.2] for the explicit description of $(*, **)$).
- (3) For any effective movable divisor D , there are a nef divisor H and $g \in \text{Bir}(M)$ such that $g^*D = H$ in $N^1(M)$. In particular, H is semi-ample and the image of $\Phi_{|mH|}$ for large $m > 0$ is a smooth projective variety, which is either $(\mathbb{P}^1)^k$ ($0 \leq k \leq n-1$) or M (see the description of the nef cone $\overline{\text{Amp}}(M)$ above).

The following proposition is crucial for Theorem 8.1:

Proposition 8.3. *Assume that M is a Calabi-Yau manifold of Wehler type of odd dimension $n = 2k+1 \geq 3$. Then, there is $f \in \text{Bir}(M)$ such that the characteristic polynomial of $f^*|N^1(M)$ is irreducible over \mathbb{Z} and it is a Salem polynomial of degree $n+1 = \text{rank } N^1(M)$. In particular, there is no non-trivial f -stable \mathbb{Q} -linear subspace of $N^1(M)_{\mathbb{Q}} := N^1(M) \otimes \mathbb{Q}$.*

Proof. By Theorem 8.2 (2), we may and will regard $\text{Bir}(M) \subset \text{O}(N^1(M)_{\mathbb{R}})$ with respect to the bilinear form $(*, **)$. By $\text{Bir}(M) = \langle \iota_k \rangle_{k=1}^{n+1}$ and the description of ι_k^* on $N^1(M)$ in Theorem 8.2 (2), the action of $\text{Bir}(M)$ on $N^1(M)_{\mathbb{R}}$ is irreducible. Thus the result follows from Theorem 3.6 (2). \square

Theorem 8.1 follows from the next proposition:

Proposition 8.4. *Let M and $f \in \text{Bir}(M)$ be as in Proposition 8.3. Then f is primitive and $d_1(f) > 1$.*

Proof. As M is a smooth Calabi-Yau manifold, f is isomorphic in codimension one. Thus $d_1(f) = r_1(f) > 1$ by Remark 2.3 (2) and by the choice of f .

We show that f is primitive. First, we observe the following:

Lemma 8.5. *There are no smooth projective variety B with a dominant rational map $\pi : M \dashrightarrow B$ of connected fibers and $f_B \in \text{Bir}(B)$ such that $\pi \circ f = f_B \circ \pi$, $0 < \dim B < \dim M$ and resolutions of general fibers of π are of Kodaira dimension 0.*

Proof. Assuming to the contrary that there are such $\pi : M \dashrightarrow B$ and $f_B \in \text{Bir}(B)$, we derive a contradiction.

Let $\nu : \tilde{M} \rightarrow M$ be a resolution of the indeterminacy $I(\pi)$ of π and $\tilde{\pi} : \tilde{M} \rightarrow B$ be the induced morphism. Let H be a very ample divisor on B . Then, by definition, $\pi^*H = \nu_*(\tilde{\pi}^*H)$ is a movable divisor on M . By Theorem 8.2 (3), there is $g \in \text{Bir}(M)$ such that $L := g^*(\pi^*H)$ is nef. Note that $|L|$ is free by the first part of Theorem 8.2 (3). So, by replacing f by $g \circ f \circ g^{-1}$ and π by $\pi \circ g^{-1}$, we may and will assume from the first that $\pi : M \dashrightarrow B$ is given by a linear subsystem Λ of a free linear system $|L|$, i.e., $\pi = \Phi_\Lambda$. In this replacement, we also note that $g \circ f \circ g^{-1}$ has the same characteristic polynomial as f on $N^1(M)$.

As $K_M = 0$ and $\kappa(\tilde{F}) = 0$ by our assumption, we have $\dim B = \kappa(M, L)$ by [Ue75, Theorem 5.11] (see also [AC13, Théorème 2.3]). As $\Lambda \subset |L|$, the morphism $\Phi_{|L|} : M \rightarrow B'$ given by the complete free linear system $|L|$ factors through π and we have a dominant rational map $q : B' \dashrightarrow B$ such that $\pi = q \circ \Phi_{|L|}$. We have then $\dim B' = \dim B$ by $\kappa(M, L) = \dim B$. Hence q is birational, as π has connected fibers.

So, by replacing π by $\Phi_{|mL|}$ for large $m > 0$, we may and will assume that $\pi : X \rightarrow B$ is a surjective morphism. Here B is smooth by Theorem 8.2 (3). Then $f_B^* = p_* \circ q^*$ is well-defined on $N^1(B)_\mathbb{Q}$ and $f_B^* N^1(B)_\mathbb{Q} = N^1(B)_\mathbb{Q}$. Here $p : \tilde{B} \rightarrow B$ is a resolution of indeterminacy of f_B and $q : \tilde{B} \rightarrow B$ is the induced morphism. Smoothness of B and \tilde{B} is used to identify one cocycles and codimension one cycles. As f is isomorphic in codimension one, it follows that $0 \neq \pi^* N^1(B)_\mathbb{Q} \subset N^1(M)_\mathbb{Q}$ is f -stable. Hence $\pi^* N^1(B)_\mathbb{Q} = N^1(M)_\mathbb{Q}$, by the irreducibility of the action of f . However, this is impossible, as $\pi^* N^1(B)_\mathbb{Q}|F = 0$ but $N^1(M)_\mathbb{Q}|F \neq 0$ for general fibers F of π . \square

From now, we closely follow an argument in [Bi16].

Lemma 8.6. *Let $P \in M$ be a general point. Then, the Zariski closure of the orbit $\{f^n(P) | n \in \mathbb{Z}\}$ is M . In particular, there is no dominant rational map $\pi : M \dashrightarrow B_1$ of connected fibers and $f_{B_1} \in \text{Bir}(B_1)$ such that $\pi_1 \circ f = f_{B_1} \circ \pi_1$, $0 < \dim B_1 < \dim M$ and resolutions of general fibers of π_1 are of general type.*

Proof. If otherwise, there are smooth projective variety C and a dominant rational map $\tau : M \dashrightarrow C$ such that $\tau \circ f = \tau$ by [AC13, Théorème 4.1]. As f is isomorphic in codimension one, it follows that $f^*v = v$ for all $v \in \tau^* N^1(C)$ by [Bi16, Lemma 4.5], a contradiction to the irreducibility of f^* on $N^1(M)$.

We show the last statement. If we would have a rational map $\pi_1 : M \dashrightarrow B_1$ as in Lemma 8.6, then $\{f_{B_1}^n(Q) | n \in \mathbb{Z}\}$ would be Zariski dense in B_1 for generic $Q \in B_1$, as so is for f and π_1 is dominant. The last statement now follows from [Bi16, Proposition]. \square

Lemma 8.7. *Assume that there are a smooth projective variety B_2 , a dominant rational map $\pi_2 : M \dashrightarrow B_2$ of connected fibers and $f_{B_2} \in \text{Bir}(B_2)$ such that $\pi_2 \circ f = f_{B_2} \circ \pi_2$ and $0 < \dim B_2 < \dim M$. Then, there are a smooth projective variety B_3 , a dominant rational map $\pi_3 : M \dashrightarrow B_3$ of connected fibers and $f_{B_3} \in \text{Bir}(B_3)$ such that $\pi_3 \circ f = f_{B_3} \circ \pi_3$, $0 < \dim B_3 < \dim M$ and resolutions of general fibers of π_3 are of Kodaira dimension 0.*

Proof. Resolutions of general fibers of π_2 are of non-negative Kodaira dimension, by the adjunction formula applied for a resolution of the indeterminacy of π_2 . So, one can consider the relative Kodaira fibration $\pi_3 : M \dashrightarrow B_3$ over B_2 and π_3 satisfies all requirement except perhaps $0 < \dim B_3 < \dim M$. We have also $0 < \dim B_3 < \dim M$, as resolutions of general fibers of π_2 are not of general type by Lemma 8.6. \square

By Lemmas 8.5 and 8.7, f has to be primitive. \square

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